Bias and Refinement of Multiscale Mean Field Models

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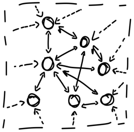
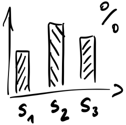
Classical Mean Field Setting

Applications
Load Balancing, Epidemics, Chemical Reactions, Networks Analysis, ...



microscopic
to macroscopic



Goal
Derive Deterministic Approximation

$$\dot{x} = f(x)$$

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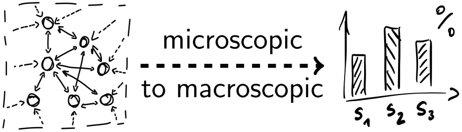


Goal
Derive Deterministic Approximation

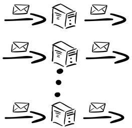
$$\dot{x} = f(x)$$

Works if particles are homogeneous & self-contained (no interaction with environment).

Justified by (asymptotic) object independence.

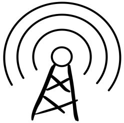
Bias Correction
Can be made more accurate using refinements which take dependencies into account. [Gast, Van Houdt]

Two Timescale Systems



Macroscopic Quantity
$\rightarrow$ slow System $X^{(N)}(t)$

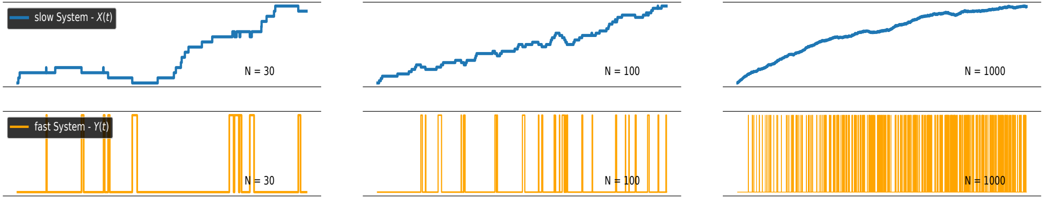
coupling
of dynamics



'Environment'
$\rightarrow$ fast System - $Y^{(N)}(t)$

Corresponding Drift

$f(x, y)$



We Study Generic Coupled Two Timescale Models

Finite set $\mathcal{T}$ of transitions:

$$(\boldsymbol{X}^{(N)}(t), \boldsymbol{Y}^{(N)}(t)) \text{ jumps to } (\boldsymbol{X}^{(N)}(t) + \ell/N, \boldsymbol{Y}') \quad \text{at rate } N \times \alpha_{\ell,\boldsymbol{y}'}(\boldsymbol{X}^{(N)}(t), \boldsymbol{Y}^{(N)}(t)).$$

Two Timescale Drift

$$F(\boldsymbol{x}, \boldsymbol{y}) := \sum_{\ell,\boldsymbol{y}' \in \mathcal{T}} \alpha_{\ell,\boldsymbol{y}'}(\boldsymbol{x}, \boldsymbol{y})\ell \in \mathbb{R}^{d_x}$$

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Common Mean Field Intuition: Use drift as dynamics for the ODE.

Problem - <u>not well defined</u> due to $\boldsymbol{y} \rightarrow \dot{\boldsymbol{x}} \overset{?}{=} F(\boldsymbol{x}, \boldsymbol{y})$

Decoupling and Averaging

Decoupled 'Fast' Dynamics

Define $K(\boldsymbol{x})$ as a transition kernel of the fast process for fixed $\boldsymbol{X} = \boldsymbol{x}$ by

$$K_{\boldsymbol{y},\boldsymbol{y}'}(\boldsymbol{x}) = \sum_{\ell} \alpha_{\ell,\boldsymbol{y}'}(\boldsymbol{x}, \boldsymbol{y}).$$

Assume $K(\boldsymbol{x})$ has a unique stationary distribution ('unichain') $\pi(\boldsymbol{x}) = (\pi_{\boldsymbol{y}}(\boldsymbol{x}))_{\boldsymbol{y} \in \mathcal{Y}}$.

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'Average' Drift $$\bar{F}(\boldsymbol{x}) := \sum_{\boldsymbol{y}} \pi_{\boldsymbol{y}}(\boldsymbol{x}) F(\boldsymbol{x}, \boldsymbol{y})$$

'Average' Mean Field Dynamics $$\dot{\boldsymbol{x}} = \bar{F}(\boldsymbol{x}).$$

Assumptions

(A_1) Finite set of transitions $\mathcal{T}$ and for all $\ell, \boldsymbol{y}' \in \mathcal{T}$
rates $\alpha_{\ell,\boldsymbol{y}'}$ are twice cont. differentiable with Lipschitz derivatives.

(A_2) $K(\boldsymbol{x})$ has a unique irreducible class for all $\boldsymbol{x} \in \mathcal{X}$.
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Additional Steady-State Assumptions

(A_3) The stochastic system has a stationary distribution denoted by $(\boldsymbol{X}_\infty^{(N)}, \boldsymbol{Y}_\infty^{(N)})$

(A_4) The ODE equilibrium point $\boldsymbol{x}(\infty)$ is unique and exponentially stable.

Results

Theorem (Steady-State)

Assume $(A_1) - (A_4)$. For all $h \in \mathcal{D}^2(\mathcal{X})$ there exists a constant C_h such that:

$$\mathbb{E}[h(\boldsymbol{X}_\infty^{(N)})] = \underbrace{h(\boldsymbol{x}(\infty))}_{\text{'Average' Mean Field}} + C_h \frac{1}{N} + o(1/N)$$

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- similar result for transient regime
- holds for $h \in \mathcal{D}(\mathcal{X} \times \mathcal{Y})$ and its averaged version too
- bias term can be computed

Refined 'Average' Mean Field

$$C_h = \sum_i \frac{\partial h}{\partial x_i}(\boldsymbol{x}(\infty))\left(V_i + T_i + S_i\right) + \frac{1}{2}\sum_{i,j}\frac{\partial^2 h}{\partial x_i x_j}(\boldsymbol{x}(\infty))\big(W_{i,j} + U_{i,j}\big)$$ - 'new' terms

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Correction Terms V, W

- closely related to [Gast, Van Houdt] 'classical' refinement terms
- error of the decoupled slow system to 'average' mean field

∗New∗ Correction Terms S, T, U

- error of decoupling of the slow system & error of 'averaging' assumption
- involved computations due to looping over 'fast' components states

Terms are **solutions to linear equations**.

$\boldsymbol{x}(\infty) + \frac{C_h}{N}$ – **Refined 'Average' Mean Field**

Proof Ideas

- **generator comparison** of stochastic and deterministic process
- use **two Poisson equations** to characterize
  - the difference of the stochastic drift and its average version

$$L_{\text{fast}} G_F(\boldsymbol{x}, \boldsymbol{y}) = F(\boldsymbol{x}, \boldsymbol{y}) - \bar{F}(\boldsymbol{x})$$

  - the fluctuation of the decoupled stochastic system around $\boldsymbol{x}(\infty)$

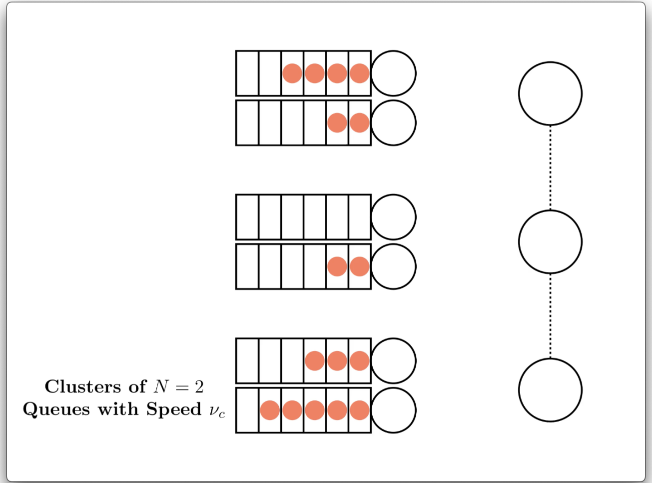
$$\Lambda G_h(\boldsymbol{x}) = h(\boldsymbol{x}) - h(\boldsymbol{x}(\infty))$$

- use equations to obtain **derivative bounds** and deduce **computable bias expressions**

Example - Random Access Network w. Interference

Model from [Cecchi et al.]:

- 'average' mean field is asymptotically exact.
- closed form solution of $\pi(\boldsymbol{x})$ available.



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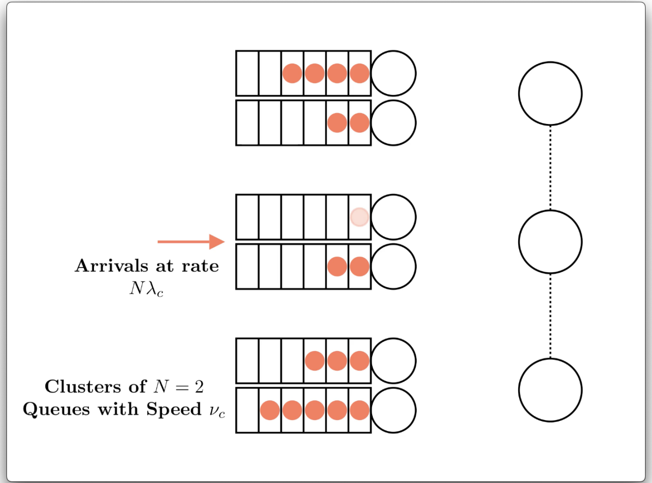

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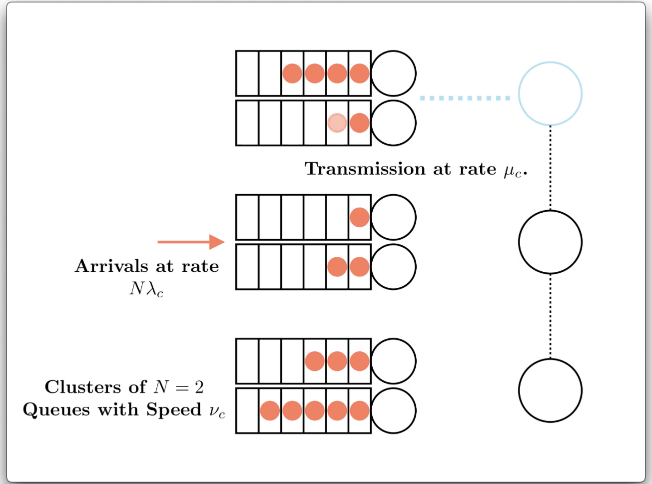
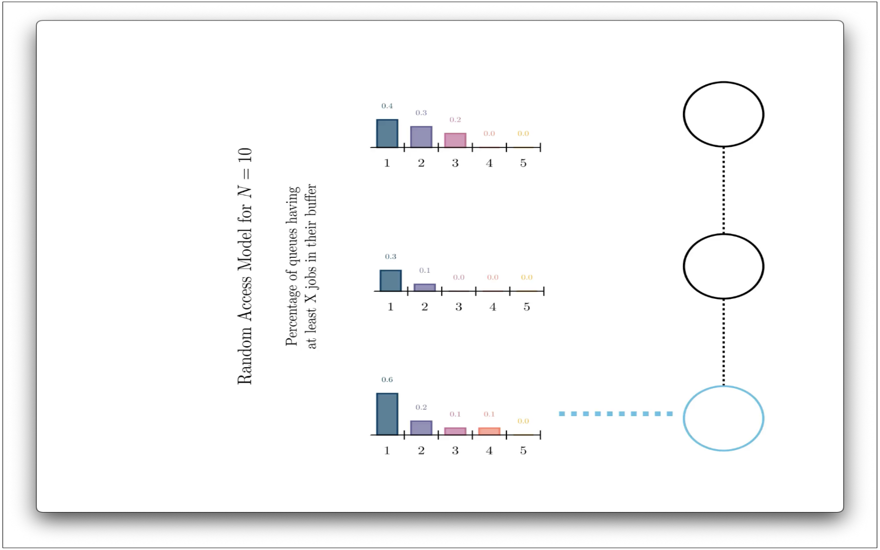


CSMA Linear 3 Node Model - Video Illustration



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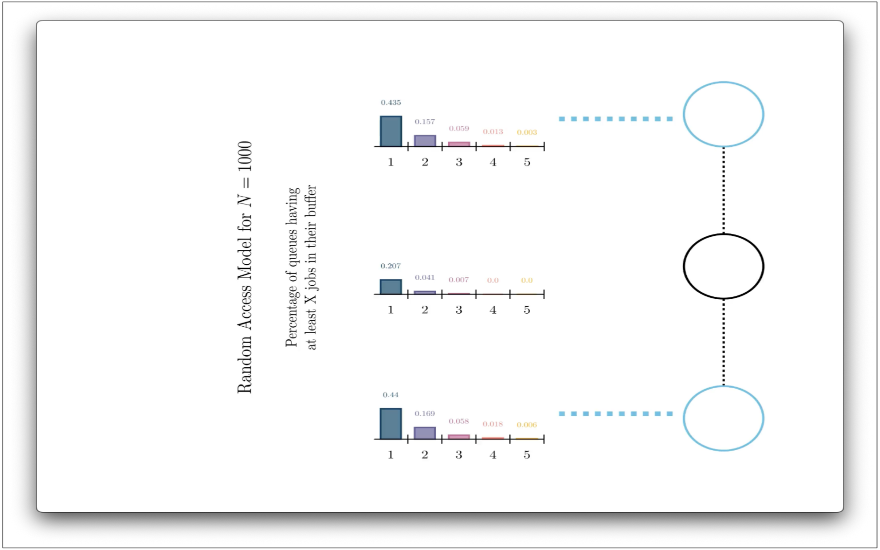
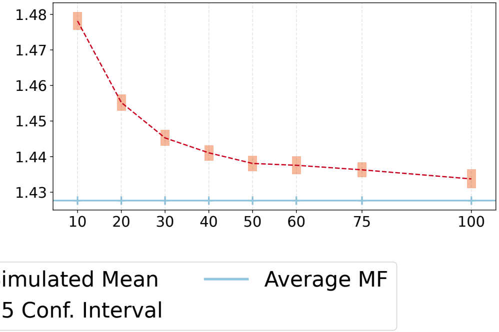


Numerical Results - Steady-State



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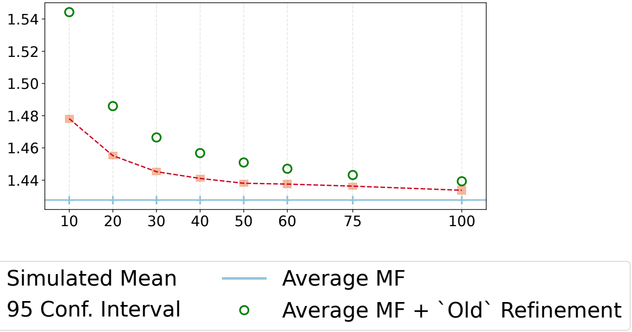

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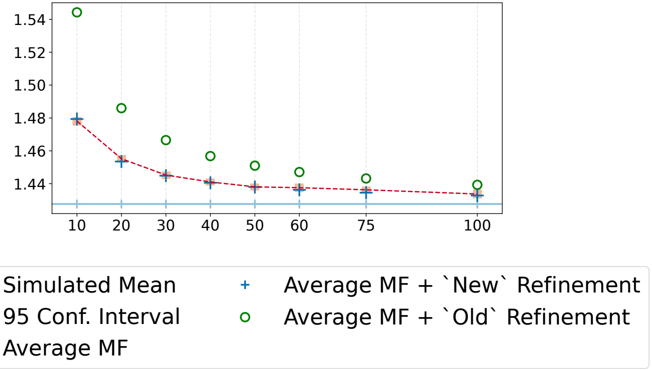

Takeaways

The ‘average’ mean field technique

- can be applied to two timescale model with increasing **accuracy of order** $O(1/N)$ in transient regime and steady-state
- can be refined in steady-state new **expansion terms**
- expansion terms can be computed efficiently through ODE and linear equations
- small hidden constants in practice

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Thank you!

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References

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